ISSN 0027-1349, Moscow University Physics Bulletin, 2012, Vol. 67, No. 1, pp. 139–142. © Allerton Press, Inc., 2012.
Original Russian Text © N.V. Gubina, V.Ch. Zhukovskii, K.G. Klimenko, S.G. Kurbanov, 2012, published in Vestnik Moskovskogo Universiteta. Fizika, 2012, No. 1, pp. 000–000.

Pion and Chiral Density Waves in a (1 + 1)-Dimensional Nambu–Jona–Lasinio Model

N. V. Gubina[a], V. Ch. Zhukovskii[a], K. G. Klimenko[b], and S. G. Kurbanov[a]
[a] *Faculty of Physics, Moscow State University, Moscow, 119991 Russia*
[b] *Institute of High Energy Physics and Dubna University (Protvino Branch), Moscow oblast, 142281 Russia*
e-mail: gubinanadya@gmail.ru, zhukovsk@phys.msu.ru, kklim@ihep.ru, serdar.kurbanov@gmail.ru
Received September 21, 2011; in final form, October 5, 2011

Abstract—A (1 + 1)-dimensional massless Nambu–Jona–Lasinio model is investigated in the limit of a large number of colors. The model describes a system with two quark flavors if μ baryon and μ_I isospin chemical potentials occur. The question of whether spatially inhomogeneous chiral and pion condensates can form in a dense quark environment is also examined.

Keywords: Nambu–Jona–Lasinio (NJL) model, chemical potential, thermodynamic potential, phase diagram, spatially inhomogeneous condensate, chiral density wave, pion-density wave
DOI: 10.3103/S0027134912010067

INTRODUCTION

In the present work we examine a method for generalizing the Gross–Neveu model [1] with one type (flavor) of quark to the case of two different flavors by considering the respective potentials. The Lagrangian of a (1 + 1)-dimensional mass model with an additional four-fermionic interaction of the Nambu–Jona–Lasinio type that describes the dense quark environment in terms of u and d quarks can be written as follows

$$\mathcal{L} = \bar{q}[\gamma^\rho i\partial_\rho + \mu\gamma^0 + \nu\tau_3\gamma^0 - m_0]q + \frac{G}{N_c}[(\bar{q}q)^2 + (\bar{q}i\gamma^5\vec{\tau}q)^2], \quad (1)$$

where the gamma matrixes are chosen in the following way: $\gamma^0 = \sigma_2$; $\gamma^1 = i\sigma_1$; $\gamma^5 = \gamma^0\gamma^1 = \sigma_3$ and two-component Dirac spinors $q(x) \equiv q_{i\alpha}(x)$ are the doublets over flavors ($i = 1, 2$) with respective Pauli matrixes τ_a ($a = 1, 2, 3$) and with N_c-let over colors ($\alpha = 1, ..., N_c$). The chemical potential μ is responsible for the nonzero baryon density of the quark environment, while the isospin chemical potential $\mu_I = 2\nu$ is examined for the case of a nonzero isospin density that considers environment asymmetry over isotopic composition.

This model is invariant with respect to the following groups of symmetry. Under $\mu_I = 0$ and $m_0 = 0$, in addition to the global $SU(N_c)$ symmetry over color index, the initial Lagrangian is invariant with respect to chiral group $SU_L(2) \times SU_R(2)$ transformation. But under $\mu_I \neq 0$ the group of symmetry is reduced to $U_{I_3L}(1) \times U_{I_3R}(1)$, where $I_3 = \tau_3/2$ is the third component of isospin operator. Let us point out that the initial Lagrangian is invariant with respect to evenness transformation.

1. THERMODYNAMIC POTENTIAL OF THE MODEL

In investigating the model, we introduce the new fields during bosonization

$$\sigma(x) = -2\frac{G}{N_c}(\bar{q}q);$$

$$\pi_a(x) = -2\frac{G}{N_c}(\bar{q}i\gamma^5\tau_a q), \quad (a = 1, 2, 3).$$

After respective substitution in the first order of decomposition under high N_c we obtain the following expression for efficient action $\mathcal{S}_{\text{eff}}(\sigma, \pi_a)$

$$\mathcal{S}_{\text{eff}}(\sigma, \pi_a) = -N_c\int d^2x\left[\frac{\sigma^2 + \pi_a^2}{4G}\right] + \tilde{\mathcal{S}}_{\text{eff}}. \quad (2)$$

The quark field contribution to efficient action is determined by the $\tilde{\mathcal{S}}_{\text{eff}}$ member in Eq. (2) and can be written in explicit form

$$\exp(i\tilde{\mathcal{S}}_{\text{eff}}) = N'\int[d\bar{q}][dq]\exp\left(i\int\{\bar{q}[\gamma^\rho i\partial_\rho + \mu\gamma^0 + \nu\tau_3\gamma^0 - \sigma - m_0 - i\gamma^5\pi_a\tau_a]q\}d^2x\right).$$

The value of mean vacuum $\langle\sigma(x)\rangle$ and $\langle\pi_a(x)\rangle$ of the generated boson fields, which are determined according to extremum condition for efficient action in a dense environment, where $\mu \neq 0$, $\mu_I \neq 0$, can depend nontrivially on x. In particular, in the present work we use an ansatz, which can be called a pion density wave

$$\langle\sigma(x)\rangle + m_0 = M, \quad \langle\pi_3(x)\rangle = 0,$$
$$\langle\pi_1(x)\rangle + i\langle\pi_2(x)\rangle = \Delta e^{2i\vec{b}\vec{x}}, \qquad (3)$$
$$\langle\pi_1(x)\rangle - i\langle\pi_2(x)\rangle = \Delta e^{-2i\vec{b}\vec{x}},$$

where M, b, Δ are constant and are the points of global minimum for thermodynamic potential (TDP) $\Omega(M, b, \Delta)$, which is determined via efficient action by the known method

$$\Omega(M, b, \Delta)\int d^2x = -\frac{1}{N_c}\mathcal{S}_{\text{eff}}\{\sigma(x), \pi_a(x)\}\big|_{\sigma(x) = \langle\sigma(x)\rangle, \pi_a(x) = \langle\pi_a(x)\rangle}.$$

During calculations by considering the chosen ansatz, it is necessary to perform the turn

$$q = (\psi_u, \psi_d)^T \longrightarrow \chi = (\chi_u, \chi_d)^T = (\psi_u e^{ibx}, \psi_d e^{-ibx})^T = e^{i\tau_3 bx}q. \qquad (4)$$

By considering transformation (4), the thermodynamic potential can be written as follows

$$\Omega(M, b, \Delta) = \frac{(M-m_0)^2 + \Delta^2}{4G} + \frac{i}{(2\pi)^2}\int d^2p \ln\det D, \qquad (5)$$

where $D = (\not p - M + \mu\gamma_0 + \nu\tau_3\gamma_0 + \tau_3\gamma_1 b - i\Delta\tau_1\gamma_5)$. The roots of equation $\det D = 0$, which are determined analytically, $\eta^{(i)} = p_0^{(i)} + \mu$, $(i = \overline{1, 4})$, present the model spectrum; after this the expression for thermodynamic potential is reduced as follows

$$\Omega(M, b, \Delta) = \frac{(M-m_0)^2 + \Delta^2}{4G} - \frac{1}{4\pi}\int_{-\infty}^{\infty} dp_1\{|\eta^{(1)} - \mu| + |\eta^{(2)} - \mu| + |\eta^{(3)} - \mu| + |\eta^{(4)} - \mu|\}. \qquad (6)$$

To overcome the ultraviolet divergence, in the last expression it is possible to use regularization (which is traditional for Gross–Neveu and Nambu–Jona–Lasinio models) by introducing a cutting for Λ (the detailed procedure for normalizing the mass model of the Gross–Neveu type was presented in [2, 3])

$$\Omega(M, b, \Delta) = \lim_{\Lambda\to\infty}\left\{\Omega_{\text{reg}}(M, b, \Delta)|_{G\to G(\Lambda)} + \frac{\Lambda^2}{\pi}\right\}. \qquad (7)$$

But the thermodynamic potential obtained by means of such a procedure depends nonphysically on the wave vector b under a zero amplitude Δ and is unlimited from below over b. The following value, which is obtained after additional subtraction, is the true thermodynamic potential of the system $\Omega^{\text{phys}}(M, b, \Delta)$ (see [4])

$$\Omega^{\text{phys}}(M, b, \Delta) = \Omega(M, b, \Delta) - \Omega(M, b, 0) + \Omega(M, 0, 0). \qquad (8)$$

The cause of the revealed loss of physical sense in the regularization of Eq. (6) is as follows: the regularization with symmetrical cutting over impulses (7) is used and this problem is discussed in [5, 6, 7].

The phase structure of the model with Lagrangian (1) under $m_0 = 0$ has been studied properly for many particular cases; for example, the process of homogenous condensate formation was examined in [8] and in [5] the existence conditions for a phase with a chiral density wave were discussed. To solve the problem of the basic state structure, we choose (among all possible forms of spatially heterogonous solutions (chiral crystals, spirals, etc. see [9–11]) the condensates in the form of a chiral density wave (CDW) (see below) and of a pion density wave (3) (PDW).

2. CDW AND PDW PHASES FOR THE MASSLESS CASE: $M_0 = 0$

Phase structure for homogenous case: $b = 0$. To study the phase portrait of the examined model for the case of homogenous condensates, we write Eq. (5) for $b = 0$ as follows

$$\eta^{(1),(2),(3),(4)} = \pm\mathscr{E}_\Delta^\pm, \quad \mathscr{E}^\pm = E \pm \nu, \quad E = \sqrt{p_1^2 + M^2}.$$

We use (7) and obtain the re-normalized expression for thermodynamic potential

$$\Omega(M, \Delta) = V_0(\sqrt{M^2 + \Delta^2}) - \int_0^\infty \frac{dp_1}{\pi}\left\{\mathscr{E}_\Delta^+ + \mathscr{E}_\Delta^- - 2\sqrt{p_1^2 + M^2 + \Delta^2}\right\}$$
$$- \int_0^\infty \frac{dp_1}{\pi}\{(\mu - \mathscr{E}_\Delta^+)\theta(\mu - \mathscr{E}_\Delta^+) + (\mu - \mathscr{E}_\Delta^-)\theta(\mu - \mathscr{E}_\Delta^-)\},$$

where

$$V_0(M) \equiv \lim_{\Lambda\to\infty}\left\{\frac{M^2}{4G(\Lambda)} - \frac{2}{\pi}\int_0^\Lambda dp_1\sqrt{p_1^2 + M^2} + \frac{\Lambda^2}{\pi}\right\} = \frac{M^2}{2\pi}\left[\ln\left(\frac{M^2}{M_0^2}\right) - 1\right],$$

and M_0 is a free parameter equal to the dynamic mass for quarks in a vacuum. The results of numerical investigation of the obtained expression for the case of homogenous chiral condensate are presented in Fig. 1 (a more detailed investigation was presented in [8]).

CDW Phase. In this case the form of the thermodynamic potential is studied (in contrast to (3)) with the use of another ansatz, in particular,

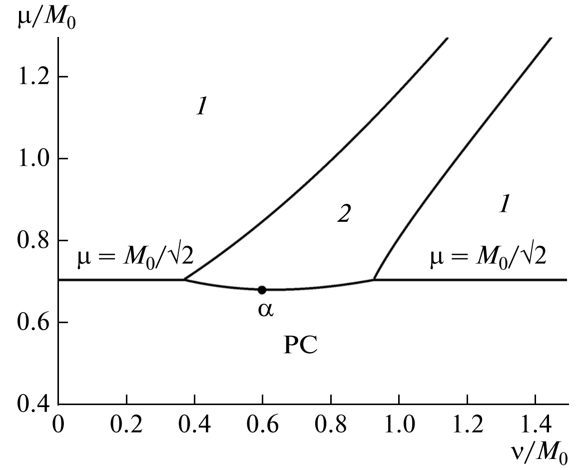

Fig. 1. A phase portrait for the case of spatially homogenous condensates. *1* is a symmetrical phase with massless quarks; *2* is a phase with mass quarks, PC is the phase of a charged pion condensation; α is the lowest point of phase 2 with respective values of $\mu_\alpha \approx 0.69M_0$ and $\nu_\alpha \approx 0.6M_0$

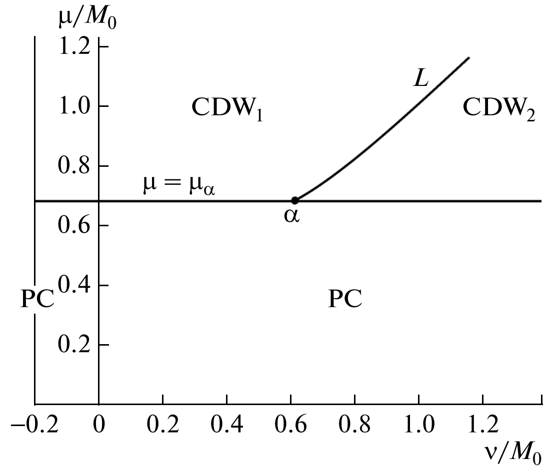

Fig. 2. A phase portrait for the model with chiral density wave. For CDW_1 (CDW_2) phase $b > 0$ ($b < 0$). Curve L ($b = 0$) corresponds to the homogenous phase with spoiled chiral symmetry. The same phase (PC) occurs at the interval $0 < \mu < \mu_\alpha \approx 0.69M_0$ at μ axis; α is the critical interphase point with coordinates $\mu = 0.69M_0$, $\nu_\alpha \approx 0.6M_0$

$$\sigma(x) = M\cos(2bx), \quad \pi_3(x) = M\sin(2bx),$$
$$\pi_1(x) = \Delta, \quad \pi_2(x) = 0.$$

The form of the chiral turn is also different: $q_w = \exp(i\gamma^5\tau_3 bx)q$. We will omit the details of the calculations here; however, they were given in [5]. The resulted thermodynamic potential can be written as follows

$$\Omega^{\text{phys}}(M, b, \Delta) = V_0(\sqrt{M^2+\Delta^2})$$
$$- \lim_{\Lambda\to\infty}\left\{\int_0^{\Lambda}\frac{dp_1}{\pi}[E_\Delta^+ + E_\Delta^- - 2\sqrt{p_1^2+M^2+\Delta^2}]\right\}$$
$$- \int_0^{\infty}\frac{dp_1}{\pi}\{(\mu - E_\Delta^+)\theta(\mu - E_\Delta^+)$$
$$+ (\mu - E_\Delta^-)\theta(\mu - E_\Delta^-)\} + \frac{(b+\nu)^2}{\pi},$$

where the $\theta(x)$ Heaviside function and model spectrum are used

$$E_\Delta^\pm = \sqrt{(E^\pm)^2 + \Delta^2},$$
$$E^\pm = E \pm (b+\nu), \quad E = \sqrt{p_1^2 + M^2}.$$

As it is seen from the phase portrait presented in Fig. 2, if we choose just this ansatz, two phases of a chiral density wave, viz., CDW_1 and CDW_2, and a homogenous pion phase can form. It turns out that CDW phases are preferable to the homogenous chiral phase and even to the vacuum phase.

Phase with Pion Density Wave. From the phase diagram presented in Fig. 3, which was obtained when we investigated the thermodynamic potential (8) and slot equation for variables M, Δ and b for the solution in the form of pion density wave (3), it is seen that there is a critical point μ_α such that if $\mu < \mu_\alpha$ there is the only homogenous pion condensate and if $\mu > \mu_\alpha$ the material is in the form of a pion density wave, excluding a thin strip for the phase of normal quark matter near line $\mu = \nu$. The width of this strip is only $0.03M_0$; it finishes in the critical point α, where $\mu = \mu_\alpha = 0.069M_0$, $\nu = 0.6M_0$.

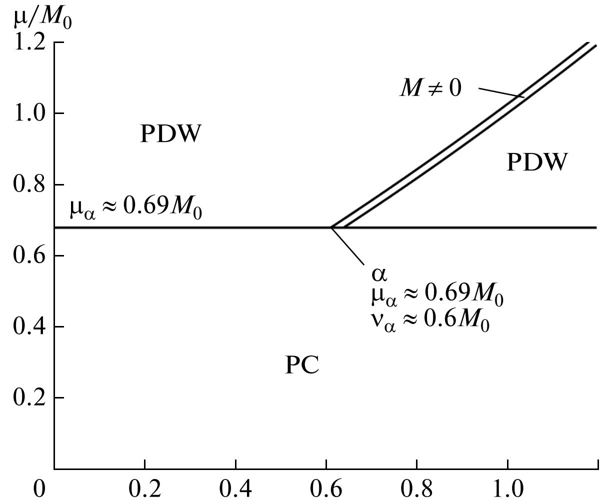

Fig. 3. A phase portrait for the model with pion density wave. PC is a phase of homogenous pion condensate ($\mu < \mu_\alpha$), PDW is the heterogeneous phase with pion density wave (under $\mu > \mu_\alpha$). The area of the plot designated as M_0 corresponds to homogenous chiral asymmetrical phase. The width of this area is ~ $0.03M_0$; α is a critical interphase point

CONCLUSIONS

In this paper a (1 + 1)-dimensional massless Nambu–Jona–Lasinio model with chemical potentials of two types was examined at a zero temperature. It was revealed that for baryon chemical potential $\mu < \mu_c$ and arbitrary nonzero values of μ_I only the phase of a charged pion condensate is formed. The spatially heterogeneous forms of chiral and pion condensates in the form of density waves with the given form are formed if $\mu > \mu_{\text{critical}}$. In this case, the results of numerical calculations do not allow us to give preference to any phase: CDW or PDW, since in different segments of space they appear with equal probability. Due to this fact, it is important to study the problem of whether hidden symmetry exists in the model and to separate the 2D case and increase the number of measurements.

REFERENCES

1. D. J. Gross and A. Neveu, Phys. Rev. D: Part. Fields **10**, 3235 (1974).
2. A. Barducci, R. Casalbuoni, M. Modugno, and G. Pettini, Phys. Rev. D: Part. Fields **51**, 3042 (1995).
3. V. Ch. Zhukovskii, K. G. Klimenko, and T. G. Khunzhura, Moscow Univ. Phys. Bull. **65**, 21 (2010).
4. C. F. Mu, L. Y. He, and Y. X. Liu, Phys. Rev. D: Part. Fields **82**, 056006 (2010).
5. D. Ebert, N. V. Gubina, K. C. Klimenko, et al., Phys. Rev. D: Part. Fields **84**, 025004 (2011).
6. E. Nakano and T. Tatsumi, Phys. Rev. D: Part. Fields **71**, 114006 (2005).
7. D. Nickel, Phys. Rev. D: Part. Fields **80**, 074025 (2009).
8. D. Ebert and K. G. Klimenko, arXiv:0902.1861.
9. S. Carignano, D. Nickel, and M. Buballa, Phys. Rev. D: Part. Fields **82**, 054009 (2010).
10. G. Basar, G. V. Dunne, and M. Thies, Phys. Rev. D: Part. Fields **79**, 105012 (2009).
11. G. Basar and G. V. Dunne, J. High Energy Phys. **1101**, 127 (2011); arXiv:1007.1962.

SPELL OK