ISSN 0027-1349, Moscow University Physics Bulletin, 2010, Vol. 65, No. 1, pp. 21–26. © Allerton Press, Inc., 2010.
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Pion Condensation in the Gross–Neveu Model

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Received October 6, 2009; in final form October 18, 2009

Abstract—The phase structure of the 2-dimensional Gross–Neveu model is studied in variables of the quark number and isospin chemical potentials at zero temperature in the limit of a large number of field components $N_c \longrightarrow \infty$.

Key words: Gross–Neveu model, quark matter.
DOI: 10.3103/S0027134910010054

INTRODUCTION

Recently, much attention has been paid to studying phase transitions in hadron matter, taking barion and isotopic chemical potentials into account. The latter is connected with experiments on collisions of heavy ions, and also with the physics of compact stars with neutrons and protons entering their structure in an unsymmetrical manner. Since the interaction in such situations is strong, it is necessary to apply nonperturbation methods and, in particular, effective models of the Nambu–Jona–Lasinio (NJL) type [1]. While the NJL models are applicable at low energies and densities, the models of the Gross—Neveu (GN) type in space with the dimension (1 + 1) [2–4] do not require these restrictions. Moreover, they effectively simulate the properties of quantum chromodynamics (QCD), such as renormalization, asymptotic freedom, and dimensional transmutation. In the limit of a large number of field components ($N_c \longrightarrow \infty$), the theorem that states that the impossibility of the spontaneous breaking of a continuous symmetry in the (1 + 1) GN model does not hold [11, 12]. Therefore, the use of the GN model for studying nonperturbation phenomena within this model at $N_c \longrightarrow \infty$ is considerably easier (see, e.g., [7, 8, 13]). Hence, it seems to be convenient to study such effects as color superconductivity and spontaneous breaking of the isotopic symmetry characteristic for realistic conditions in (3 + 1)-dimensional space using the example of the GN model in the limit of large N_c numbers.

Unlike our previous publication [13], where a similar model was used to study a system of zero-mass quarks with two aromas in a finite volume at the finite isotopic chemical potential $\mu_I \neq 0$ and zero barion density $\mu = 0$, in this study both chemical potentials and the mass of quarks are considered to be nonzero, and space has a standard topology $R^1 \times R^1$. Thus, we assume that condensates are spatially homogeneous (the case of inhomogeneous condensates at $\mu_I = 0$ has been recently considered in [9, 10, 14, 15]).

1. THE MODEL AND ITS THERMODYNAMIC POTENTIAL

Let us consider a (1 + 1)-dimensional model of the dense quark matter consisting of zero-mass quarks of two different aromas (u and d quarks), described by a Lagrangian

$$L = \bar{q}\left[\gamma^\nu i\partial_\nu - m_0 + \mu\gamma^0 + \frac{\mu_I}{2}\tau_3\gamma^0\right]q + \frac{G}{N_c}[(\bar{q}q)^2 + (\bar{q}i\gamma^5\vec{\tau}q)^2]. \quad (1)$$

This model presents a generalization of the GN model [2], since here a two-component Dirac spinor of a quark field $q(x) \equiv q_{i\alpha}(x)$ also represents a doublet on aromas ($i = 1,2$ or $i = u, d$) and an N_c multiplet on colors ($\alpha = 1, ..., N_c$) (in Eq.(1)). The summation over aroma, color, and spinor indices is meant; τ_k (k = 1, 2, 3) are Pauli matrices, μ is the chemical potential for the quark number, and μ_I is the chemical potential for the isospin number). At $\mu_I = 0$, $m_0 = 0$, Lagrangian Eq. (1) is invariant with respect to the transformations of the chiral groups $SU_L(2) \times SU_R(2)$. At $\mu_I \neq 0$, $m_0 = 0$ the symmetry is lowered to the group $U_{I_3L}(1) \times U_{I_3R}(1)$ ($I_3 = \tau_3/2$) is the third component of the isospin operator and the indices L and R denote the left- or right-hand subgroup). This symmetry can also be presented as a product of subgroups $U_{I_3}(1) \times U_{AI_3}(1)$, where $U_{I_3}(1)$ is an isospin subgroup and $U_{AI_3}(1)$ is an axial isospin subgroup, which transform quarks as $q \longrightarrow \exp(i\alpha\tau_3)q$ and $q \longrightarrow \exp(i\alpha\gamma^5\tau_3)q$, respectively.

In the case of $m_0 \neq 0$, $\mu_I = 0$, Lagrangian Eq. (1) is invariant with respect to the group $SU_I(2)$ representing a diagonal subgroup of the chiral group $SU_L(2) \times SU_R(2)$. In the most general case, where $m_0 \neq 0$, $\mu_I \neq 0$, the initial model Eq. (1) is symmetric with respect to the above isospin subgroup $U_{I_3}(1)$. In all the above particular cases, this model is invariant with respect to the color group $SU(N_c)$.

Linearizing the model in Eq. (1) by introducing the composite boson fields $\sigma(x)$ and $\pi_a(x)$ ($a = 1, 2, 3$),

$$\sigma(x) = -2\frac{G}{N_c}(\bar{q}q), \quad \pi_a(x) = -2\frac{G}{N_c}(\bar{q}i\gamma^5\tau_a q), \tag{2}$$

we obtain an equivalent Lagrangian

$$\tilde{L} = \bar{q}\left[\gamma^\nu i\partial_\nu - m_0 + \mu\gamma^0 + \frac{\mu_I}{2}\tau_3\gamma^0 - \sigma - i\gamma^5\pi_a\tau_a\right]q - \frac{N_c}{4G}[\sigma\sigma + \pi_a\pi_a]. \tag{3}$$

As follows from Eq. (2), the boson fields are subject to the transformations of the isospin subgroup $U_{I_3}(1)$: $\sigma \longrightarrow \sigma$; $\pi_3 \longrightarrow \pi_3$; $\pi_1 \longrightarrow \cos(2\alpha)\pi_1 + \sin(2\alpha)\pi_a$; $\pi_2 \longrightarrow \cos(2\alpha)\pi_2 - \sin(2\alpha)\pi_1$. At $N_c \longrightarrow \infty$, the main term of the expansion of the thermodynamic potential (TDP) of the model on $1/N_c$ is

$$\Omega_{\mu,\nu}(M, \Delta) = \frac{(M - m_0)^2 + \Delta^2}{4G} + i\int\frac{d^2p}{(2\pi)^2}\ln\{[(p_0 + \mu)^2 - (E_\Delta^+)^2][(p_0 + \mu)^2 - (E_\Delta^-)^2]\}, \tag{4}$$

where $E_\Delta^\pm = \sqrt{(E^\pm)^2 + \Delta^2}$, $E^\pm = E \pm \nu$, $\nu = \mu_I/2$ and $E = \sqrt{p_1^2 + M^2}$, $M = m_0 + \langle\sigma\rangle$, $\Delta = \langle\pi_1\rangle$. It is clear that the TDP diverges in the ultraviolet. We introduce $|p_1| < \Lambda$ and then, according to the properties of the 2-dimensional GN model, the bonding constant G and the mass of the "naked" quark m_0 become functions of Λ. The procedure for the renormalization of the massive GN model is well-known (see, for example, [5, 6, 16, 17]). Following this procedure, for $G \equiv G(\Lambda)$ we write

$$\frac{1}{2G(\Lambda)} = \frac{1}{\pi}\int_{-\Lambda}^{\Lambda}dp_1\frac{1}{\sqrt{M_0^2 + p_1^2}} = \frac{2}{\pi}\ln\left(\frac{\Lambda + \sqrt{M_0^2 + \Lambda^2}}{M_0}\right) \tag{5}$$

and introduce the new free finite renormalization parameter m ($m_0 = mG(\Lambda)$) independent of Λ. As a result, we obtain the following finite renormalized TDP:

$$\Omega_{\mu,\nu}(M, \Delta) = V_0(M, \Delta) - \frac{mM}{2} - \int_{-\infty}^{\infty}\frac{dp_1}{2\pi}\Big\{E_\Delta^+ + E_\Delta^- - 2\sqrt{p_1^2 + M^2 + \Delta^2} + (\mu - E_\Delta^+)\theta(\mu - E_\Delta^+) + (\mu - E_\Delta^-)\theta(\mu - E_\Delta^-)\Big\}, \tag{6}$$

where

$$V_0(M, \Delta) = \frac{M^2 + \Delta^2}{2\pi}\left[\ln\left(\frac{M^2 + \Delta^2}{M_0^2}\right) - 1\right] = \Omega_{\mu,\nu}(M, \Delta)|_{\mu = 0, \nu = 0, m = 0}. \tag{7}$$

Since for a system with a strong interaction parity in a vacuum should be conserved we assume $\Delta = 0$ in Eq. (7). Then the global TDP minimum Eq. (7) appears at the point $M = M_0$, i.e., the dynamic mass generated in a vacuum represents the parameter M_0 introduced in Eq. (5).

2. THE PHASE STRUCTURE OF THE MODEL

a) The particular case: $\mu = 0$, $\mu_I = 0$, $m \neq 0$.

We introduce the α parameter as follows: $m \equiv \alpha M_0/\pi$. Then from Eq. (6) we obtain the TDP

$$\Omega_0(M, \Delta) = V_0(M, \Delta) - \frac{\alpha M_0 M}{2\pi}. \tag{8}$$

From here we obtain the gap equations:

$$\frac{\partial\Omega_0(M, \Delta)}{\partial M} = \frac{\partial\Omega_0(M, \Delta)}{\partial\Delta} = 0, \tag{9}$$

where

$$\frac{\partial\Omega_0(M, \Delta)}{\partial M} = 2M\ln\left(\frac{M^2 + \Delta^2}{M_0^2}\right) - \alpha M_0, \qquad \frac{\partial\Omega_0(M, \Delta)}{\partial\Delta} = 2M\ln\left(\frac{M^2 + \Delta^2}{M_0^2}\right). \tag{10}$$

The system Eq. (9) has several solutions, but the global minimum point (GMP) of the potential Eq. (8) corresponds to the value $\Delta = 0$. Then the first equation in Eqs. (9) with respect to M has three solutions of different signs. The largest one corresponds to the GMP of the potential. This gap M is shown in Fig. 1 as a function of α. Since the densities of the quark number n_q and the isospin number n_I are zero in the GMP, the ground state of the model at $\mu = 0$ and $\mu_I = 0$ corresponds to a vacuum and therefore the gap M presents the dynamic quark mass in a vacuum. In the chiral limit $\alpha = 0$, the gap M apparently coincides with M_0. In addition, Fig. 1 shows the behavior of the π-meson mass as a function of α at $\mu = 0$ and $\mu_I = 0$. It appears

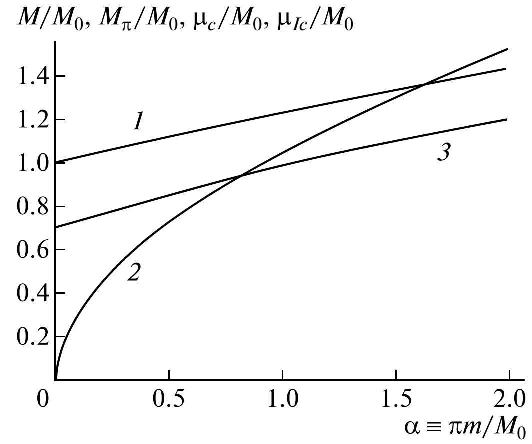

Fig. 1. Dynamic quark mass M (curve *1*) and π-meson mass M_π (curve *2*) as functions of $\alpha \equiv \pi m/M_0$ at $\mu = 0$, $\mu_I = 0$. Curve *3*, the critical value μ_c for the vacuum – the normal phase transition of the quark matter (at $\mu_I = 0$); the critical value μ_{I_c} of the vacuum – the pion condensate transition is shown by curve *2*, i.e., $\mu_{I_c} = M_\pi$.

that M_π coincides with the critical value μ_{Ic} of the isotopic chemical potential μ_I at which the system transfers from the vacuum state to the pion condensate phase. This is shown in Fig. 1. In this figure, the behavior of the critical value μ_c of the chemical potential μ as a function of α, at which the system transfers from the vacuum state to the normal phase of quark matter at $\nu = 0$ (see below), is shown as well.

The ratio of M to the pion mass M_π in vacuum (at $\mu = 0$ and $\mu_I = 0$) has to correspond to real physics. Therefore, further in the most general case (at $\mu \neq 0$ and $\nu \neq 0$) we use the ratio usually assumed in the (3 + 1)-dimensional NJL model when studying the dense quark matter (at $\mu = 0$ and $\mu_I = 0$) [18]: $M =$ 350 MeV and $M_\pi = 140$ MeV, i.e., $M/M_\pi = 5/2$. Then, according to Fig. 1, this choice corresponds to the value $\alpha = \alpha_0 \approx 0.17$, that gives $M/M_0 \approx 1.04$, $M_\pi/M_0 \approx 0.42$, and $m/M_0 \approx 0.05$, where M_0 is the dynamic quark mass in the zero-mass GN model at $\mu = 0$ and $\mu_I = 0$.

b) The particular case: $\mu \neq 0$, $\mu_I = 0$.

From the relation Eq. (6) we obtain the following expression for the TDP at $\mu \neq 0$, $\mu_I = 0$:

$$\Omega_\mu(M, \Delta) = V_0(M, \Delta) - \frac{\alpha M_0 M}{2\pi} + \frac{\theta(\mu - \sqrt{M^2 + \Delta^2})}{\pi}\Omega'_\mu(M, \Delta), \tag{11}$$

where

$$\Omega'_\mu(M, \Delta) = (M^2 + \Delta^2)\ln\left(\frac{\mu + \sqrt{\mu^2 - M^2 - \Delta^2}}{\sqrt{M^2 + \Delta^2}}\right) - \mu\sqrt{\mu^2 - M^2 - \Delta^2}. \tag{12}$$

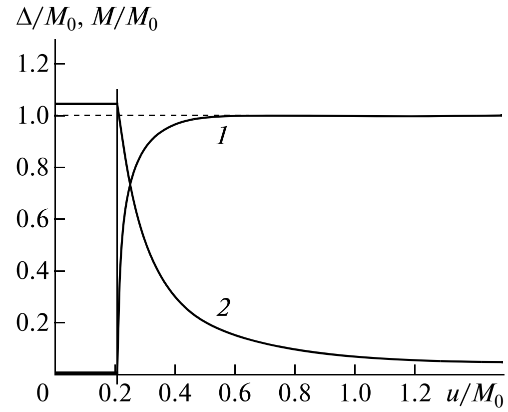

Fig. 2. Dynamic quark mass M as a function of μ at $\mu_I = 0$ and $\alpha = \alpha_0 \approx 0.17$. Here $\mu_c/M_0 = 0.76$.

From the gap equation for the TDP Eq. (11) it follows that in the GMP $\Delta = 0$ and M satisfies the equation

$$\theta(\mu^2 - M^2)\ln\frac{(\mu + \sqrt{\mu^2 - M^2})^2}{M_0^2} + \theta(M^2 - \mu^2)\ln\frac{M^2}{M_0^2} = \frac{\alpha M_0}{2M}. \tag{13}$$

One can see that at $\mu < \mu_c$ the GMP is located at $(M, \Delta = 0)$, and the critical μ_c value and the gap M are shown in Fig. 1. The system is in the vacuum state $n_q = 0$ and $n_I = 0$. At $\mu > \mu_c$ a phase of the normal quark matter with the finite density n_q is formed; however, at $\mu_I = 0$ its isospin density is zero and $n_I = 0$. In the particular case when $\alpha = \alpha_0 \approx 0.17$, the behavior of the gap M in the GMP is given in Fig. 2, where $\mu_c = 0.76M_0$.

c) The particular case: $\mu = 0$, $\mu_I \neq 0$.

At $\mu = 0$, the TDP is determined by the formula Eq. (6), in which the second line is absent, since the θ-functions in this case are zero:

$$\Omega_\nu(M, \Delta) = V_0(M, \Delta) - \frac{mM}{2} - \int_{-\infty}^{\infty}\frac{dp_1}{2\pi}\left\{E_\Delta^+ + E_\Delta^- - 2\sqrt{p_1^2 + M^2 + \Delta^2}\right\}. \tag{14}$$

This expression can be presented in the integrated form if one uses the elliptic integrals

$$\mathbf{I}_0 = \int_0^\infty \frac{du}{\sqrt{Au^3 + Bu^2 + Cu + 1}}, \qquad \mathbf{I}_{-1} = \int_{\frac{1}{\Lambda}}^\infty \frac{du}{u\sqrt{Au^3 + Bu^2 + Cu + 1}}, \tag{15}$$

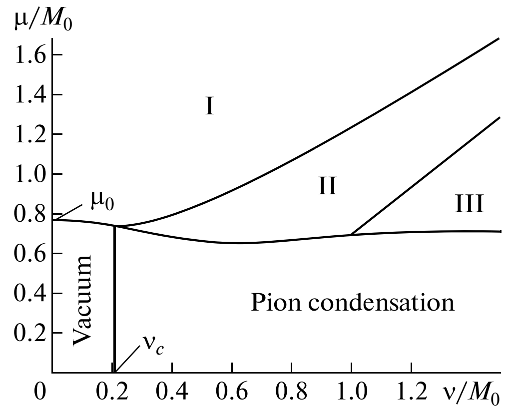

Fig. 3. Gaps Δ (curve *1*) and M (curve *2*) as functions of $\nu \equiv \mu_I/2$ in the case $\mu = 0$ and $\alpha = \alpha_0 \approx 0.17$.

$$\mathbf{I}_1 = \int_0^{\Lambda} \frac{u du}{\sqrt{Au^3 + Bu^2 + Cu + 1}},$$

where

$$A = 2M((M+\nu)^2 + \Delta^2),$$
$$B = 5M^2 + 6M\nu + \nu^2 + \Delta^2, \quad (16)$$
$$C = 4M + 2\nu,$$

with the change of the sign $\nu \longrightarrow -\nu$ leading to the replacement of the A, B and C parameters by

$$A' = 2M((M-\nu)^2 + \Delta^2),$$
$$B' = 5M^2 - 6M\nu + \nu^2 + \Delta^2, \quad (17)$$
$$C' = 4M - 2\nu,$$

and the corresponding replacement of the integrals Eq. (15) $\mathbf{I}_0$, $\mathbf{I}_{-1}$, and $\mathbf{I}_1$ by $\mathbf{I}'_0$, $\mathbf{I}'_{-1}$ and $\mathbf{I}'_1$. Then Eq. (14) is

$$\Omega_\nu(M, \Delta) = V_0(M, \Delta) - \frac{mM}{2}$$
$$- \frac{M}{4\pi}[A(\nu\mathbf{I}_1 + \mathbf{I}_0) - A'(\nu\mathbf{I}'_1 - \mathbf{I}'_0)] \quad (18)$$
$$+ \frac{(M^2 + \Delta^2)}{2\pi} \times (\mathbf{I}_{-1} + \mathbf{I}'_{-1}).$$

This expression is finite in the limit $\Lambda \longrightarrow \infty$ in spite of the divergence of separate summands. Hence the gap equation follows:

$$\frac{\partial \Omega_\nu(M, \Delta)}{\partial M} = \frac{\partial \Omega_\nu(M, \Delta)}{\partial \Delta} = 0. \quad (19)$$

Note that in the (3 + 1)-dimensional NJL model with the pion condensate in the case of the nonzero naked (current) quark mass [19], at a critical value of the isospin chemical potential μ_{Ic} being the pion mass $\mu_{Ic} = M_\pi$ in vacuum at $\mu = 0$ and $\mu_I = 0$, the continuous phase transition of the second kind from the vacuum phase (possible at $\nu < \nu_c = M_\pi/2$) with $M(\nu) \equiv M(0) \neq 0$, $\Delta(\nu) = 0$ to the pion condensate phase (at $\nu > \nu_c$), where $M(\nu) \neq 0$, $\Delta(\nu) \neq 0$ occurs. The GMP ($M(\nu)$, $\Delta(\nu)$) with following properties: $M(\nu) \longrightarrow M(\nu_c) \equiv M(0)$, $\Delta(\nu) \longrightarrow 0$ if $\nu \longrightarrow \nu_{c+}$ corresponds to this phase. Here we also use the notations $\nu = \mu_I/2$, and $M(0)$ for the dynamic quark mass in vacuum.

A similar picture of the pion condensation occurs in the framework of the massive GN model as well. In fact, the numerical study of the TDP Eq. (6) at $\mu = 0$ shows that at the critical point ν_c the phase transition of the second kind from vacuum to the pion condensation phase (i.e., the GMP is a continuous function of ν at the critical point $\nu = \nu_c$) occurs. To determine ν_c and to show that the equality $\nu_c = M_\pi/2$ holds as well in the massive GN model, it is necessary to note that at $\nu > \nu_c$ the coordinates ($M(\nu)$, $\Delta(\nu)$) the GMP of the TDP satisfies Eqs. (19). Since at $\nu = \nu_c$ a continuous phase transition takes place, i.e., $\Delta(\nu_c) = 0$, $M(\nu_c) \equiv M(0)$,[1] at the critical point $\nu = \nu_c$ this pair of equalities is transformed as follows:

$$\alpha M_0 = 2M(0)\ln\frac{M^2(0)}{M_0^2}, \quad (20)$$

$$\ln\frac{M^2(0)}{M_0^2}$$
$$= 2\nu_c^2 \int_0^{\infty} dp_1 \frac{1}{\sqrt{p_1^2 + M^2(0)}(p_1^2 + M^2(0) - \nu_c^2)}. \quad (21)$$

By using Eq. (21), one finds from Eq. (20)

$$\frac{\alpha M_0}{2M(0)} = 2\nu_c^2 \int_0^{\infty} dp_1 \frac{1}{\sqrt{p_1^2 + M^2(0)}(p_1^2 + M^2(0) - \nu_c^2)}. \quad (22)$$

It is easy to show that the π mass M_π in a vacuum satisfies the same relation and then $\nu_c = M_\pi/2$, i.e., the critical value μ_{Ic} is the π mass M_π at $\mu = 0$ and $\mu_I = 0$ at the arbitrary α values. As a result, the plots μ_{Ic} and M_π as functions of α are shown as a single curve in Fig. 1.

Thus, at $\nu < \nu_c$ there is a phase corresponding to the empty space with $n_q = n_I = 0$, i.e., vacuum. In a vacuum, $\Delta = 0$, but the gap $M \neq 0$ and does not depend on ν (its α dependence is shown in Fig. 1). At $\nu > \nu_c$, the pion condensation phase with $n_q = 0$ and $n_I \neq 0$ is implemented in the model. In this phase, both gaps M and Δ are nonzero and depend on ν. The isospin symmetry $U_{I_3}(1)$ is spontaneously broken in this phase. For a certain value of the parameter $\alpha = \alpha_0 \approx 0.17$, the ν dependence of gaps is shown in Fig. 3, where $\nu_c \approx 0.21M_0$.

d) The general case: $\mu \neq 0$, $\mu_I \neq 0$.

By differentiating the TDP Eq. (6), we obtain the following gap equations:

[1] The α function of $M(0)$ presents the gap M shown in Fig. 1 as curve *1*.

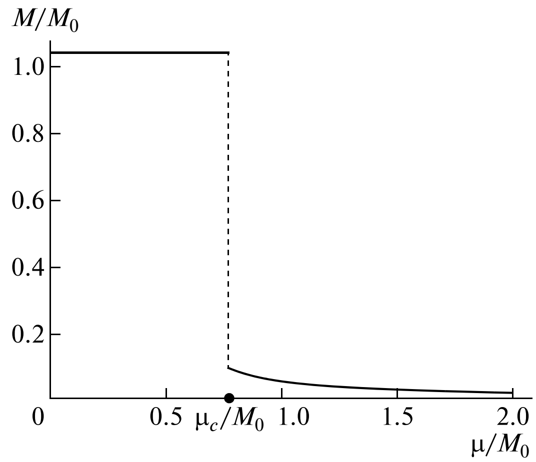

Fig. 4. Phase picture of the model in the variables μ and $\nu \equiv \mu_I/2$ at $\alpha = \alpha_0 \approx 0.17$. Here $\nu_c/M_0 \approx 0.21$, $\mu_c/M_0 \approx 0.76$. All curves correspond to the phase transitions of the first kind, except for the border between the vacuum and the pion condensate phase.

$$\frac{\partial\Omega_{\mu,\nu}(M,\Delta)}{\partial M} = \frac{\partial\Omega_{\mu,\nu}(M,\Delta)}{\partial\Delta} = 0, \tag{23}$$

where

$$\frac{\partial\Omega_{\mu,\nu}(M,\Delta)}{\partial M} = \frac{\partial\Omega_{\nu}(M,\Delta)}{\partial M} + \frac{M}{\pi}\int_0^\infty dp_1\left\{\frac{E^+\theta(\mu-E_\Delta^+)}{EE_\Delta^+} - \frac{E^-\theta(\mu-E_\Delta^-)}{EE_\Delta^-}\right\}, \tag{24}$$

$$\frac{\partial\Omega_{\mu,\nu}(M,\Delta)}{\partial\Delta} = \frac{\partial\Omega_{\nu}(M,\Delta)}{\partial\Delta} + \frac{\Delta}{\pi}\int_0^\infty dp_1\left\{\frac{\theta(\mu-E_\Delta^+)}{E_\Delta^+} - \frac{\theta(\mu-E_\Delta^-)}{E_\Delta^-}\right\}. \tag{25}$$

The phase picture constructed on the basis of these equations, at $\alpha = \alpha_0 \approx 0.17$, is shown in Fig. 4. One can see a vacuum, the pion condensate phase, and three normal phases of quark matter I, II, and III.

In the pion condensation phase, the gaps Δ and M are nonzero and hence the isospin symmetry $U_{I_3}(1)$ is spontaneously broken. In this phase, the gaps do not depend on μ; however, they strongly depend on ν (see Fig. 3). At the points (ν, μ) in other phases in Fig. 4, the Δ coordinate of the GMP of the TDP is zero, i.e., the isospin symmetry $U_{I_3}(1)$ is conserved there. At the same time, the M coordinate is nonzero. In the vacuum phase, the gap M does not depend on (ν, μ), i.e., it is constant, namely, $M \approx 1.04M_0$ at $\alpha = \alpha_0 \approx 0.17$. Since the gap continuously depends on μ and ν on the border between the vacuum phase and the pion condensate phase, one can conclude that a transition of the second kind occurs between these phases.

CONCLUSIONS

In this study it is shown that the phase of the charged pion condensation is implemented in the noncompact region of the change of the chemical potentials: $\mu_I > M_\pi$ does not exceed $M_0/\sqrt{2}$, where M_π is the π-meson vacuum. In this phase, the isospin symmetry $U_{I_3}(1)$ is spontaneously broken and zero-mass Goldstone boson excitations appear. All one-quark excitations in this phase have a gap. As a result, the density of the quark number n_q disappears in the pion condensation phase. The same properties of this phase are predicted in the framework of some parametrizations of the NJL model (see, for example, [20–22]). At the same time, in the phase diagram of the NJL model, the pion condensation phase occupies a compact region, and in some parametrization schemes gapless pion condensation can be formed [19–22].

At comparatively large values of the chemical potential of the quark number we found a rather large variety of the normal phases of quark matter I, II, and III (see Fig. 4) in which n_q is nonzero. In phase I, both u- and d-quarks are gapless quasiparticles; however, in phases II and III only the u-quarks are gapless, while the d-quarks have a gap. Thus, such dynamic effects in the dense quark matter as the transfer phenomena (for example, conductivity) can occur in qualitatively different manners in phases I, II, and III. A more realistic model for studying the phase diagrams of quantum chromodynamics should include a finite temperature and inhomogeneous condensates [14, 15].

ACKNOWLEDGMENTS

The authors are grateful to Professor D. Ebert for useful discussion of the results of this study. The study is supported by DAAD (German Academic Exchange Service).

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